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Efficacy of a New Geometric Stiffness Matrix for Buckling Load Analyses

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Abstract: This paper investigates the development and performance of a new higher-order geometric stiffness matrix that more closely approximates the theoretically derived stiffness coefficients. Factors that influence the accuracy of the solution are studied using two columns, two braced frames, and one unbraced frame. Discussion is provided when the new geometric stiffness matrix can be used to improve the buckling load analysis results and when it may provide only nominal additional benefit.

Key words: Geometric stiffness matrix, buckling load, stability functions, structural frame.

1. Introduction

This paper presents and assesses the performance of a new geometric stiffness matrix to potentially reduce the required number of elements per member of a linear buckling analysis. The effects of nonlinear material behavior are not considered in this study as the majority of routine building design considers only linear, elastic material behavior [1]. Assessment of the new geometric stiffness matrix is conducted on two columns and three frames with known "exact" closed-form solutions for the elastic critical buckling load [2]. The frames were modeled in the MASTAN2 [3] analysis software. The software is capable of performing a linear buckling analysis using the inverse iteration method [4]. All members were modeled as planar 6-dof line elements with elastic material behavior. Models have perfect geometries when comparing the results with the "exact" solutions.

2. New Geometric Stiffness Matrix

A nonlinear tangent stiffness matrix for a beam-column element was developed by Ekhande et al. [5] using stability functions to account for the effect of axial force on flexural stiffness. The explicit expressions for the stability functions of a planar beam-column are given in Eqs. (1)-(5).

$$\beta = L\sqrt{\frac{P}{EI}} \tag{1}$$

$$C_1 = \frac{\beta^2(1-cos\beta)}{2sin\beta\left(tan\frac{\beta}{2}-\frac{\beta}{2}\right)} \tag{2}$$

$$C_2 = \frac{\beta(sin\beta-\beta cos\beta)}{2sin\beta\left(tan\frac{\beta}{2}-\frac{\beta}{2}\right)} \tag{3}$$

$$C_3 = \frac{\beta(\beta-sin\beta)}{2sin\beta\left(tan\frac{\beta}{2}-\frac{\beta}{2}\right)} \tag{4}$$

$$C_4 = \frac{\beta^3 sin\beta}{2sin\beta\left(tan\frac{\beta}{2}-\frac{\beta}{2}\right)} \tag{5}$$

Using the geometric stiffness matrix as developed by Yang and McGuire [6] for use in an updated Lagrangian nonlinear elastic analysis, the stability functions appear in the global stiffness matrix for a planar beam element. They presented simplified 2nd-order expressions for the stability functions in their geometrical stiffness matrix K_g as given in Eqs. (6)-(9).

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These equations have been used extensively over the decades and form the basis of the original K_g in MASTAN2 [3].

$$C_1 = 6 - \frac{\beta^2}{10} \tag{6}$$

$$C_2 = 4 - \frac{2\beta^2}{15} \tag{7}$$

$$C_3 = 2 + \frac{\beta^2}{30} \tag{8}$$

$$C_4 = 12 - \frac{6\beta^2}{5} \tag{9}$$

These simplified 2nd-order expressions begin to deviate from the "exact" expressions when $\beta > 2$ as illustrated in Figs. 1-4. In order to reduce this error, while maintaining the simplicity and numerical stability of a 6th-order polynomial expression, Eqs. (10)-(13) were developed based on a nonlinear regression analysis of data produced using β increments of 0.01 (r^2 = 0.999) in Eqs. (2)-(5). These equations form the basis of the new K_g and were added to the source code of MASTAN2 [3].

$$C_1 = 6 - \frac{1{,}031\beta^2}{10{,}000} - \frac{227\beta^6}{5{,}000{,}000} \tag{10}$$

$$C_2 = 4 - \frac{1{,}403\beta^2}{10{,}000} - \frac{733\beta^6}{5{,}000{,}000} \tag{11}$$

$$C_3 = 2 + \frac{743\beta^2}{20{,}000} + \frac{2\beta^6}{19{,}763} \tag{12}$$

$$C_4 = 12 - \frac{603\beta^2}{500} - \frac{466\beta^6}{5{,}128{,}205} \tag{13}$$

3. Braced Columns and Frames

The relative error is used to evaluate each modeled condition as given in Eq. (14) where P is determined from closed-form equations [2] and P_{cr} is determined from an elastic buckling analysis in MASTAN2 [3]. Using the original K_g and the new K_g in MASTAN2, Column 1 in Fig. 5 has $\beta = \pi$ in Eq. (15) and $P = \pi^2$ in Eq. (1).

$$\text{Relative Error} = \frac{P_{cr} - P}{P} \cdot 100\% \tag{14}$$

$$\sin\beta = 0 \tag{15}$$

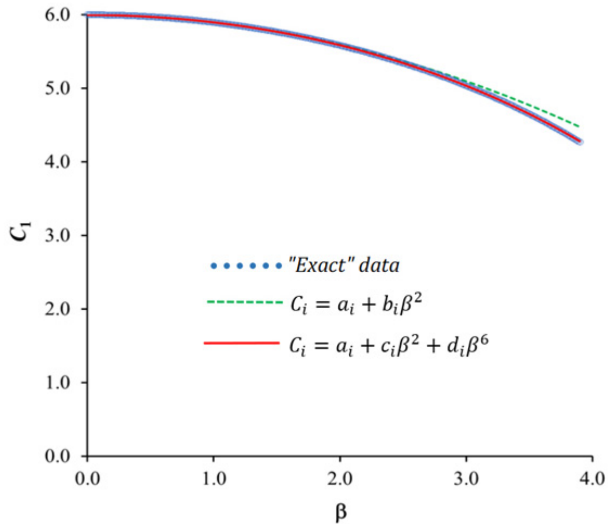

Fig. 1 2nd-order and 6th-order approximations for C_1.

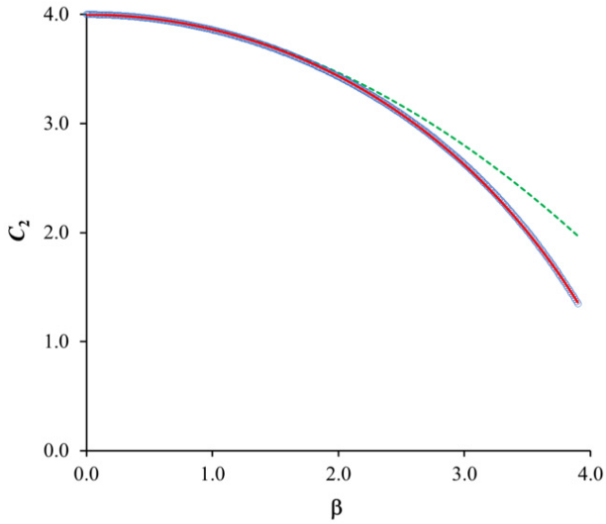

Fig. 2 2nd-order and 6th-order approximations for C_2.

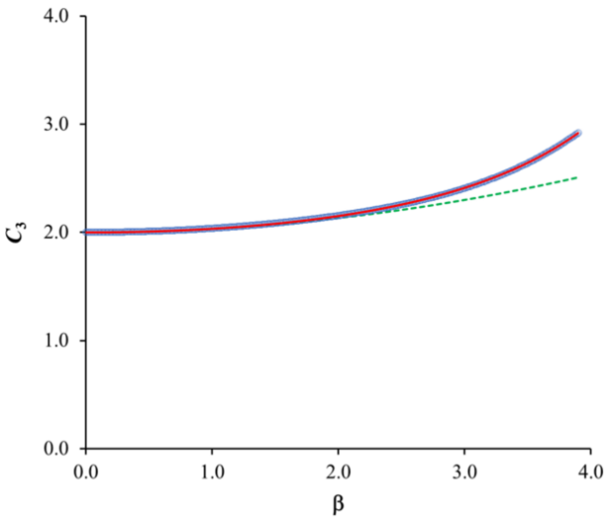

Fig. 3 2nd-order and 6th-order approximations for C_3.

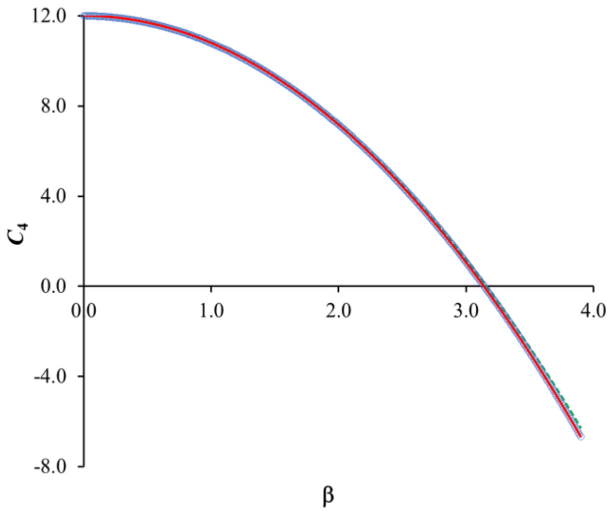

Fig. 4 2nd-order and 6th-order approximations for C_4.

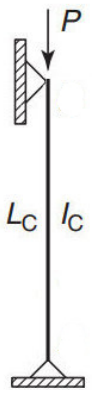

Fig. 5 Column 1.

Table 1 Column 1 analysis conditions and results.

	P_{ref}	Elem/Mem	P_{cr}	Rel. Error
Orig. K_g	1	1	12.00	21.59
	9.8696	1	12.00	21.59
	9.8696	2	9.944	0.75
	9.8696	4	9.875	0.05
	1	4	9.875	0.05
New K_g	1	1	11.26	14.09
	9.8696	1	9.921	0.52
	9.8696	2	9.820	-0.50
	9.8696	4	9.844	-0.26
	1	4	9.844	-0.26

As given in Table 1, high relative errors are produced with $P_{ref} = 1$ when using only one element to model the column, but the new expressions give a relative error of less than one percent when using only one element when $P_{ref} = P$. The old expressions give good results when two or more elements are used, and there is no P_{ref} effect when four elements are used.

Column 2 in Fig. 6 has $\beta = 4.49341$ in Eq. (16) and $P = 20.1907$ in Eq. (1).

$$tan\beta - \beta = 0 \tag{16}$$

As indicated in Table 2, high relative errors are produced with $P_{ref} = 1$ when only one element is used, but the new expressions give a relative error of less than one percent when using only one element when $P_{ref} = P$. The old expressions give comparable results only when four elements are used.

Frame 1 in Fig. 7 is used to study the effect of differing column and beam stiffnesses on the modeling results. The $\gamma = 4.6$, 8 and 24 conditions in Table 3 produce $\beta = 4.2152$, 4.32205 and 4.43275 in Eq. (18) and $P = 17.7679$, 18.6801 and 19.6493 in Eq. (1), respectively.

$$\gamma = \frac{I_B L_C}{I_C L_B} \tag{17}$$

$$tan\beta - \frac{3\gamma\beta}{\beta^2 + 3\gamma} = 0 \tag{18}$$

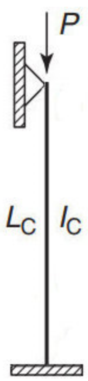

Fig. 6 Column 2.

Table 2 Column 2 analysis conditions and results.

	P_{ref}	Elem/Mem	P_{cr}	Rel. Error
Orig. K_g	1	1	30.00	48.59
	20.19	1	30.00	48.59
	20.19	2	20.71	2.58
	20.19	4	20.23	0.20
	1	4	20.23	0.20
New K_g	1	1	28.48	41.06
	20.19	1	19.99	-0.99
	20.19	2	20.15	-0.20
	20.19	4	20.11	-0.40
	1	4	20.11	-0.40

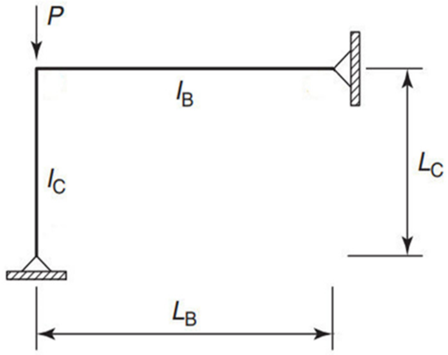

Fig. 7 Frame 1.

Table 3 Frame 1 analysis conditions and results.

	P_{ref}	γ	Elem/Mem	P_{cr}	Rel. Error
	1	4.6	1	25.64	44.31
	17.768	4.6	1	25.64	44.31
Orig. K_g	17.768	4.6	4	17.74	-0.16
	18.680	8	1	27.27	45.99
	19.649	24	1	28.95	47.33
	1	4.6	1	24.21	36.26
	17.768	4.6	1	17.56	-1.17
New K_g	17.768	4.6	4	17.65	-0.66
	18.680	8	1	18.44	-1.28
	19.649	24	1	19.37	-1.42

The new K_g gives good results when using one element per member with $P_{ref} = P$, but four elements per member are needed with the old expressions to give comparable results to those with the new expressions. As with the two columns, high relative errors occur with $P_{ref} = 1$ when using only one element per member.

Frame 2 in Fig. 8 is used to study the effect of the number of elements per member on the relative error. The $\gamma = 2/3$ condition in Table 4 produces $\beta = 3.53992$ in Eq. (20) and $P = 12.5310$ in Eq. (1).

$$c = \frac{1}{\beta^2}\left(1 - \frac{\beta}{tan\beta}\right) \quad (19)$$

$$\left(\frac{1}{c^2} + \frac{12\gamma}{c} + 24\gamma^2\right) = 0 \quad (20)$$

As with the previous examples, high relative errors occur with $P_{ref} = 1$ when using only one element per member, but there is no P_{ref} effect when four elements per member are used. The new expressions give a relative error of less than one percent when using only one element when $P_{ref} = P$, but two or more elements are needed to obtain comparable errors with the old expressions.

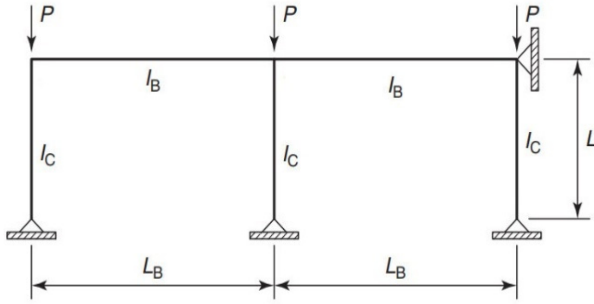

Fig. 8 Frame 2.

Table 4 Frame 2 analysis conditions and results.

	P_{ref}	γ	Elem/Mem	P_{cr}	Rel. Error
	1	0.667	1	16.24	29.61
	12.53	0.667	1	16.24	29.61
Orig. K_g	12.53	0.667	2	12.64	0.88
	12.53	0.667	4	12.53	0.00
	1	0.667	4	12.53	0.00
	1	0.667	1	15.24	21.63
	12.53	0.667	1	12.54	0.08
New K_g	12.53	0.667	2	11.95	-4.63
	12.53	0.667	4	11.89	-5.11
	1	0.667	4	11.89	-5.11

4. Unbraced Frame

The new 6th-order polynomial expressions were also studied using one unbraced frame with a known critical buckling load equation [2]. The unbraced frame in Fig. 9 was used to investigate the P_{ref} requirements and the number of elements per member that are needed to achieve accurate critical buckling load results.

Frame 3 in Fig. 3 is used to study the P_{ref} effect and the number of elements per member on the relative error. The $\gamma = 2/3$ condition in Table 5 produces $\beta = 1.29913$ in Eq. (21) and $P = 1.6877$ in Eq. (1). When using only one element per member and $P_{ref} = 1$, the relative errors are very small when using either the old or new polynomial expressions in K_g. There is also little beneficial effect to increasing the number of elements per member or to use $P_{ref} = P$ with the unbraced frame.

$$\beta^2\left(\frac{1}{c^2} + \frac{12\gamma}{c} + 24\gamma^2\right) - \frac{8\gamma}{c}\left(\frac{1}{c} + 3\gamma\right) = 0 \quad (21)$$

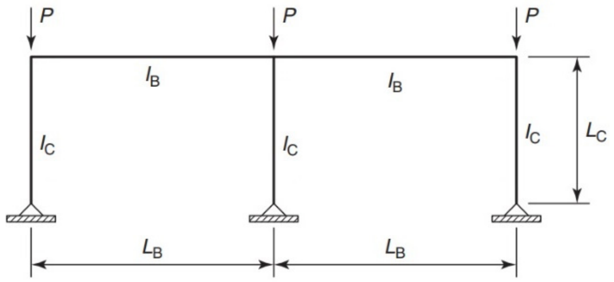

Fig. 9 Frame 3.

Table 5 Frame 3 analysis conditions and results.

	P_{ref}	γ	Elem/Mem	P_{cr}	Rel. Error
Orig. K_g	1	0.667	1	1.691	0.20
	1.6877	0.667	1	1.691	0.20
	1.6877	0.667	2	1.687	-0.04
	1.6877	0.667	4	1.687	-0.04
	1	0.667	4	1.687	-0.04
New K_g	1	0.667	1	1.691	0.20
	1.6877	0.667	1	1.682	-0.34
	1.6877	0.667	2	1.685	-0.16
	1.6877	0.667	4	1.687	-0.04
	1	0.667	4	1.687	-0.04

5. Conclusions

This paper presented a new geometric stiffness matrix with 6th-order polynomial expressions for the coefficients that more closely approximate the theoretically derived stability functions. The factors that influenced the accuracy of the solution scheme were studied using two columns, two braced frames, and one unbraced frame. It was found that the new geometric stiffness matrix gave improved results only when the structures were braced and P_{ref} was close to the critical buckling load. Under these conditions, it was found that only one element per member was needed to obtain excellent results, and this held true over a wide range of beam-to-column stiffness ratios of braced frames. The critical buckling load study of the unbraced frame revealed there was no advantage to using the new geometric stiffness matrix; this was true regardless of the number of elements used per member and the P_{ref} load.

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